

# The orbit method for unipotent groups over finite field

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According to A.A.Kirillov's orbit method there exists one to one correspondence between the irreducible representations of an arbitrary connected, simply connected Lie group and its coadjoint orbits. This correspondence between orbits and representations makes possible to solve problems of the representation theory in terms of coadjoint orbits. The orbit method initiate many papers beginning from 1962. It turns out that the ideas of the orbit method are useful for the large classes of Lie groups (see [3, 4]), and also for some matrix groups, defined over finite field.

In our paper we obtain formula for multiplicities of certain representations of unipotent groups over finite field in terms of coadjoint orbits (see theorem 9 and corollaries). For reader's convenience we formulate and prove the main statements of the orbit method over finite field (see [5]).

Let  $K = \mathbb{F}_q$  be a finite field of characteristic  $p$  having  $q = p^m$  elements. Let  $\mathfrak{g}$  be a subalgebra of the Lie algebra  $\mathfrak{ut}(N, K)$ , consisting of all upper triangular matrices with zeros on the diagonal. Suppose that  $p$  is large enough to determine the exponential  $\exp(x)$  map on  $\mathfrak{g}$ . For instance, let  $p \geq N$ . Then the exponential map is a bijection of the Lie algebra  $\mathfrak{g}$  onto the subgroup  $G = \exp(\mathfrak{g})$  of the unitriangular group  $\text{UT}(N, K)$ . One can define the adjoint representation of the group  $G$  on  $\mathfrak{g}$  by the formula  $\text{Ad}_g(x) = gxg^{-1}$ .

Denote by  $\mathfrak{g}^*$  the conjugate space of  $\mathfrak{g}$ . One can define the coadjoint representation of the group  $G$  in  $\mathfrak{g}^*$  by the formula  $\text{Ad}_g^*\lambda(x) = \lambda(\text{Ad}_g^{-1}x)$ .

Note that if  $\mathfrak{g}^\lambda$  is a stabilizer of  $\lambda \in \mathfrak{g}^*$ , then the subgroup  $G^\lambda = \exp(\mathfrak{g}^\lambda)$  is a stabilizer of  $\lambda$  in  $G$ . One can calculate the number of elements  $|\Omega|$  of the orbit  $\Omega = \text{Ad}_G^*(\lambda)$  by the formula

$$|\Omega| = \frac{|G|}{|G^\lambda|} = \frac{|\mathfrak{g}|}{|\mathfrak{g}^\lambda|} = q^{\dim \mathfrak{g} - \dim \mathfrak{g}^\lambda} = q^{\frac{1}{2} \dim \Omega}. \quad (1)$$

**Definition 1.** A subalgebra  $\mathfrak{p}$  of  $\mathfrak{g}$  is a polarization of  $\lambda \in \mathfrak{g}^*$ , if  $\mathfrak{p}$  is a maximal isotropic subspace for the skew symmetric bilinear form  $B_\lambda(x, y) = \lambda([x, y])$  on  $\mathfrak{g}$ . Recall that the subspace  $\mathfrak{p}$  is isotropic, if  $B_\lambda(x, y) = 0$  for any  $x, y \in \mathfrak{p}$ .

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Note that any polarization contains the stabilizer  $\mathfrak{g}^\lambda$ , because  $\mathfrak{p} + \mathfrak{g}^\lambda$  is an isotropic subspace.

**Proposition 2.** Any linear form  $\lambda$  on a nilpotent Lie algebra  $\mathfrak{g}$  has a polarization.

**Proof.** We shall prove using induction method for the dimension of the Lie algebra  $\mathfrak{g}$ . The statement is obvious for one dimensional Lie algebras, since in this case Lie algebra is a polarization. Assume that the statement is proved for all Lie algebras of dimension  $< \dim(\mathfrak{g})$ . We are going to prove the statement for  $\dim(\mathfrak{g})$ .

If the dimension of a center  $\mathfrak{z}$  the Lie algebra  $\mathfrak{g}$  greater than one, then one can prove existence of polarization applying induction assumption for the factor algebra of  $\mathfrak{g}$  with respect to the ideal  $\text{Ker}(\lambda|_{\mathfrak{z}})$ . Similarly, for the case  $\dim(\mathfrak{z}) = 1$ ,  $\lambda|_{\mathfrak{z}} = 0$ .

Let  $\mathfrak{z} = Kz$  and  $\lambda(z) \neq 0$ . Consider the two dimensional ideal  $Ky + Kz$ , containing  $\mathfrak{z}$ . There exists a character  $\alpha$  of the Lie algebra  $\mathfrak{g}$  such that  $\text{ad}_u(y) = \alpha(u)z$  for any  $u \in \mathfrak{g}$ . The kernel  $\mathfrak{g}_0$  of the character  $\alpha$  is an ideal of codimension one in  $\mathfrak{g}$ . There exists an element  $x \in \mathfrak{g}$  such that  $\mathfrak{g} = Kx + \mathfrak{g}_0$  and  $[x, y] = z$ .

Denote by  $\lambda_0$  the restriction of  $\lambda$  on  $\mathfrak{g}_0$ . According the induction assumption  $\lambda_0$  has a polarization  $\mathfrak{p}_0$  in  $\mathfrak{g}_0$ . Let us prove that  $\mathfrak{p}_0$  is also a polarization for  $\lambda$  in  $\mathfrak{g}$ . Really,  $\mathfrak{p}_0$  is a subalgebra and a maximal isotropic subspace in  $\mathfrak{g}_0$ ; we will show that  $\mathfrak{p}_0$  is a maximal isotropic subspace in  $\mathfrak{g}$ . Suppose that one can extend  $\mathfrak{p}_0$  to an isotropic subspace adding the element  $x + u_0$ , where  $u_0 \in \mathfrak{g}_0$ . Note that  $z, y$  belong to the stabilizer  $\mathfrak{g}_0^{\lambda_0} \subset \mathfrak{p}_0$ . Then  $0 = \lambda([x + u_0, y]) = \lambda([x, y]) = \lambda(z) \neq 0$ . A contradiction.  $\square$

**Proposition 3.** Let  $\mathfrak{p}$  be a polarization of  $\lambda \in \mathfrak{g}^*$ ,  $P = \exp(\mathfrak{p})$ ,  $\Omega(\lambda)$  be the coadjoint orbit of  $\lambda$ ,  $\pi$  be the natural projection of  $\mathfrak{g}^*$  onto  $\mathfrak{p}^*$ ,  $L^\lambda = \pi^{-1}\pi(\lambda)$ . Then

- 1)  $\dim \mathfrak{p} = \frac{1}{2} (\dim \mathfrak{g} + \dim \mathfrak{g}^\lambda)$ ;
- 2)  $|L^\lambda| = \sqrt{|\Omega(\lambda)|}$ ;
- 2)  $L^\lambda = \text{Ad}_P^* \lambda$ , in particular  $L^\lambda \subset \Omega(\lambda)$ .

**Proof.** The statement 1) follows from the formula of dimension of a maximal isotropic subspace for the skew symmetric bilinear form  $B_\lambda(x, y)$ . From 1) we obtain

$$\text{codim } \mathfrak{p} = \frac{1}{2} (\dim \mathfrak{g} - \dim \mathfrak{g}^\lambda) = \frac{1}{2} \dim \Omega(\lambda).$$

This implies the statement 2):

$$|L^\lambda| = q^{\text{codim } \mathfrak{p}} = q^{\frac{1}{2} \dim \Omega(\lambda)} = \sqrt{|\Omega(\lambda)|}.$$

Since  $\lambda([x, y]) = 0$  for any  $x, y \in \mathfrak{p}$ , we have  $\text{ad}_{\mathfrak{p}}^* \lambda(y) = 0$  for any  $y \in \mathfrak{p}$ . Then  $\text{Ad}_P^* \lambda(y) = \lambda(y)$  for any  $y \in \mathfrak{p}$ . This is equivalent to

$$\text{Ad}_P^* \lambda \subset L^\lambda.$$

The equality  $\text{Ad}_P^* \lambda = L^\lambda$  is true, since this subsets have equal number of elements:

$$|\text{Ad}_P^* \lambda| = \frac{|P|}{|G^\lambda|} = q^{\dim \mathfrak{p} - \dim \mathfrak{g}^\lambda} = q^{\frac{1}{2} \dim \Omega(\lambda)} = |L^\lambda|. \square$$

Fix a non trivial character  $e^x : K \rightarrow \mathbb{C}^*$ . We have

$$\sum_{t \in \mathbb{F}_q} e^{\alpha t} = \begin{cases} q, & \alpha = 0, \\ 0, & \alpha \neq 0. \end{cases} \quad (2)$$

The equality (2) is easy to prove: the image of homomorphism  $e^x$  is a subgroup of  $\mathbb{C}^*$ ; if  $\alpha \neq 0$ , then this subgroup is nontrivial and coincides with the subgroup of all roots of some order  $m \neq 1$  of unity; the sum of all roots of order  $m \neq 1$  of unity equals to zero.

Restriction of  $\lambda$  on its polarization  $\mathfrak{p}$  defines a character (one dimensional representation)  $\xi$  of the group  $P = \exp(\mathfrak{p})$  by the formula

$$\xi_\lambda(\exp(x)) = e^{\lambda(x)}.$$

Consider the induced representation

$$T^\lambda = \text{ind}(\xi_\lambda, P, G). \quad (3)$$

Denote by  $\chi_\lambda(g) = \text{Tr } T^\lambda(g)$  the character of representation  $T^\lambda$ .

**Theorem 4.**

$$\chi_\lambda(g) = \frac{1}{\sqrt{|\Omega|}} \sum_{\mu \in \Omega(\lambda)} e^{\mu(\ln(g))}$$

**Proof.** Extend  $\xi_\lambda$  from  $P$  to  $G$  by the formula

$$\tilde{\xi}_\lambda(u) = \begin{cases} \xi_\lambda(u) & , \text{ if } u \in P, \\ 0 & , \text{ if } u \notin P. \end{cases}$$

Formula (2) implies

$$\tilde{\xi}_\lambda(u) = \frac{1}{|L^\lambda|} \sum_{\mu \in L^\lambda} \xi_\mu(u) = \frac{1}{|L^\lambda|} \sum_{p \in P} \xi_{\text{Ad}_p^*(\lambda)}(u).$$

Choose the system of representatives  $\{g_i : i = \overline{1, k}\}$  of the classes  $G/P$ . Using the well known formula for induced characters (see [6, chapter 6]), we

obtain

$$\chi_\lambda(u) = \sum_{g_i^{-1}ug_i \in P} \tilde{\xi}_\lambda(g_i^{-1}ug_i) = \frac{1}{|L^\lambda|} \sum_{i=1, p \in P}^k \xi_{\text{Ad}_p^* \lambda}(g_i^{-1}ug_i) = \frac{1}{|L^\lambda|} \sum_{i=1, p \in P}^k \xi_{\text{Ad}_{g_i p}^* \lambda}(u),$$

Finally,

$$\chi_\lambda(u) = \frac{1}{|L^\lambda|} \sum_{g \in G} \xi_{\text{Ad}_g^* \lambda}(u) = \frac{1}{\sqrt{|\Omega|}} \sum_{\mu \in \Omega(\lambda)} e^{\mu(\ln(u))}. \square$$

**Theorem 5.**

- 1)  $\dim T^\lambda = q^{\frac{1}{2} \dim \Omega(\lambda)} = \sqrt{|\Omega|}$ .
- 2) The representation  $T^\lambda$  does not depend on the choice of polarization.
- 3) The representation  $T^\lambda$  is irreducible.
- 4) Representations  $T^\lambda$  and  $T^{\lambda'}$  are equivalent if and only if  $\lambda$  and  $\lambda'$  belong to the same coadjoint orbit.
- 5) For any irreducible representation  $T$  of the group  $G$  there exists  $\lambda \in \mathfrak{g}^*$  such that the representation  $T$  is equivalent to  $T^\lambda$ .

**Proof.** The statement 1) follows from

$$\dim T^\lambda = \dim(\text{ind}(\xi_\lambda, P, G)) = q^{\text{codim } \mathfrak{p}}.$$

The statement 2) is a corollary of the theorem 4.

Let us show that the system of characters  $\{\chi_\lambda\}$ , where  $\lambda$  is running through some system of representatives of the coadjoint orbits, is orthonormal. Let  $\Omega, \Omega'$  be two coadjoint orbits and  $\lambda, \lambda'$  be representatives of this orbits. Then

$$\begin{aligned} (\chi_\lambda, \chi_{\lambda'}) &= \frac{1}{|G|} \sum_{u \in G} \chi_\lambda(u) \overline{\chi_{\lambda'}(u)} = \frac{1}{|G|} \cdot \frac{1}{\sqrt{|\Omega|} \cdot \sqrt{|\Omega'|}} \cdot \sum_{\mu \in \Omega, \mu' \in \Omega', u \in G} \xi_\mu(u) \overline{\xi_{\mu'}(u)} = \\ &= \frac{1}{|G|} \cdot \frac{1}{\sqrt{|\Omega|} \cdot \sqrt{|\Omega'|}} \cdot \sum_{\mu \in \Omega, \mu' \in \Omega', u \in G} e^{(\mu - \mu') \ln(u)}. \end{aligned}$$

Applying

$$\sum_{x \in \mathfrak{g}} e^{\eta(x)} = \begin{cases} |G| & , \text{ if } \eta = 0, \\ 0 & , \text{ if } \eta \neq 0 \end{cases},$$

we obtain that, if  $\Omega \neq \Omega'$ , then  $(\chi_\lambda, \chi_{\lambda'}) = 0$ .

In the case  $\Omega = \Omega'$ , we have got

$$(\chi_\lambda, \chi_\lambda) = \frac{1}{|G| \cdot |\Omega|} \sum_{\mu, \mu' \in \Omega} \sum_{x \in \mathfrak{g}} e^{(\mu - \mu')x} = \frac{1}{|G| \cdot |\Omega|} \cdot |\Omega| \cdot |G| = 1.$$

This proves 3) and 4).

We shall use notation  $T^\Omega$  for the class of equivalent representations  $T^\lambda$ , where  $\lambda \in \Omega$ . To prove statement 5) we verify that the sum of squares of dimensions of irreducible representations  $\{T^\Omega : \Omega \in \mathfrak{g}^*/G\}$  equals to the number of elements of the group:

$$\sum_{\Omega \in \mathfrak{g}^*/G} (\dim T^\Omega)^2 = \sum_{\Omega \in \mathfrak{g}^*/G} \left( \sqrt{|\Omega|} \right)^2 = \sum_{\Omega \in \mathfrak{g}^*/G} |\Omega| = |\mathfrak{g}^*| = |G|.$$

This proves 5).  $\square$

**Lemma 6.** Let  $\mathfrak{g}$  be a nilpotent Lie algebra,  $\mathfrak{g}_0$  be a subalgebra of  $\mathfrak{g}$  of codimension one. Then  $\mathfrak{g}_0$  is an ideal of  $\mathfrak{g}$ .

**Proof.** Suppose the contrary. Then  $[\mathfrak{g}, \mathfrak{g}_0] \neq \mathfrak{g}_0$ ; there exist the elements  $y \in \mathfrak{g}_0$ ,  $x \notin \mathfrak{g}_0$  such that  $[x, y] = \alpha x \bmod \mathfrak{g}_0$ ,  $\alpha \neq 0$ . The subalgebra  $\mathfrak{g}_0$  is invariant with respect to  $\text{ad}_y$ . Since  $\mathfrak{g} = kx \oplus \mathfrak{g}_0$ , the operator  $\text{ad}_y$  is not nilpotent in  $\mathfrak{g}/\mathfrak{g}_0$ ; this contradicts to assumption that the Lie algebra  $\mathfrak{g}$  is nilpotent.  $\square$

**Lemma 7.** Let  $\mathfrak{g}, \mathfrak{g}_0$  be as in lemma 6,  $\pi$  is a projection  $\mathfrak{g}^* \rightarrow \mathfrak{g}_0^*$ ;  $\lambda_0 \in \mathfrak{g}_0^*$ ,  $\omega = \text{Ad}_{G_0}^*(\lambda_0)$ ,  $\mathfrak{g}^{\lambda_0} = \{x \in \mathfrak{g} : \lambda_0([x, \mathfrak{g}_0]) = 0\}$ .

1) Let the subalgebra  $\mathfrak{g}^{\lambda_0}$  belong to  $\mathfrak{g}_0$ . Then

1a)  $\pi^{-1}(\lambda_0)$  lie in the same  $\text{Ad}_G^*$ -orbit  $\Omega$ ;

1b)  $\dim \Omega = \dim \omega + 2$  (i.e.  $|\Omega| = q^2|\omega|$ ).

2) Let the subalgebra  $\mathfrak{g}^{\lambda_0}$  do not lie in  $\mathfrak{g}_0$ . Then for any  $\text{Ad}_G^*$ -orbit  $\Omega$ , which has nonempty intersection with  $\pi^{-1}(\lambda_0)$ , the projection  $\pi$  establishes one to one correspondence between  $\Omega$  and  $\omega$ ; in particular,  $\dim \Omega = \dim \omega$ .

**Proof.**

1) Suppose that the subalgebra  $\mathfrak{g}^{\lambda_0}$  belongs to  $\mathfrak{g}_0$ . Since  $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g}_0$ , the formula  $B_0(x, y) = \lambda_0([x, y])$  defines a skew symmetric bilinear form on the Lie algebra  $\mathfrak{g}$ . The kernel  $V$  of the bilinear form  $B_0$  coincides with  $\mathfrak{g}^{\lambda_0}$  and belongs to  $\mathfrak{g}_0$ . The kernel  $V_0$  of the restriction  $B_0$  on  $\mathfrak{g}_0$  coincides with  $\mathfrak{g}_0^{\lambda_0}$ .

Let us prove that there exists a pair of elements  $u \in \mathfrak{g} \setminus \mathfrak{g}_0$  and  $v \in V_0$ , such that  $B_0(u, v) = 1$ . Really, decompose  $\mathfrak{g}_0 = L_0 \oplus V_0$ , where  $L_0$  is a subspace, with the property that the restriction of bilinear form  $B_0$  on  $L_0$  is nondegenerate. Choose an arbitrary element  $u' \in \mathfrak{g} \setminus \mathfrak{g}_0$  and consider the linear form  $B_0(u', \cdot)$  on  $L_0$ . There exists  $x_0 \in L_0$  such that  $B_0(u', \cdot) = B_0(x_0, \cdot)$  on  $L_0$ . The element  $u = u' - x_0$  satisfies  $B_0(u, L_0) = 0$ . Since  $u \notin V$ , there exists  $v \in V_0$  such that  $B_0(u, v) = 1$ .

By direct calculations, we verify that for any  $\lambda \in \pi^{-1}(\lambda_0)$  the following

equalities are valid

$$\begin{cases} \text{Ad}_{\exp(tv)}^* \lambda(u) = \lambda(u) + t, \\ \text{Ad}_{\exp(tv)}^* \lambda(y) = \lambda(y) \quad \text{for any } y \in \mathfrak{g}_0. \end{cases} \quad (4)$$

This implies the statement 1a).

The orbit  $\Omega$  is a union

$$\Omega = \bigcup_{t \in K} \pi^{-1}(\omega_t), \quad (5)$$

where  $\omega_t = \text{Ad}_{\exp(tu)}^* \omega$  is a coadjoint orbit in  $\mathfrak{g}_0^*$ . Let us show that the orbits  $\omega_t$  are pairwise different. Really, if not, there exists  $t' \neq t'' \in K$  and  $g'_0, g''_0 \in G_0$  such that

$$\text{Ad}_{\exp(t'u)}^* \text{Ad}_{g'_0}^* \lambda_0 = \text{Ad}_{\exp(t''u)}^* \text{Ad}_{g''_0}^* \lambda_0.$$

Then

$$\text{Ad}_{\exp(tu)}^* \text{Ad}_{g_0}^* \lambda_0 = \lambda_0,$$

where  $t = t' - t'' \in K^*$  and  $g_0$  is an element of  $G_0$ . Then the stabilizer  $\exp(\mathfrak{g}^{\lambda_0})$  does not belong to  $G_0$ ; this contradicts to assumption of the item 1). Using (5), we obtain  $|\Omega| = q^2 |\omega|$ . Therefore  $\dim \Omega = \dim \omega + 2$ . This proves 1b).

Turn to proof of the item 2). Suppose that the subalgebra  $\mathfrak{g}^{\lambda_0}$  does not belong to  $\mathfrak{g}_0$ . For an arbitrary nonzero element  $x$  of  $\mathfrak{g}^{\lambda_0} \setminus \mathfrak{g}_0$  we have decomposition  $\mathfrak{g} = Kx \oplus \mathfrak{g}_0$ . The group  $G$  is a semidirect product  $G = G_0 X$ , where  $X = \{\exp(tx) : t \in K\}$ .

Let  $\lambda \in \pi^{-1}(\lambda_0)$ . The equality

$$\lambda([x, \mathfrak{g}]) = \lambda([x, Kx \oplus \mathfrak{g}_0]) = \lambda_0([x, \mathfrak{g}_0]) = 0$$

implies that  $x$  belongs to  $\mathfrak{g}^\lambda$ . The subgroup  $X$  lies in the stabilizer of  $\lambda$ . The orbit  $\Omega(\lambda)$  coincides with  $\text{Ad}_{G_0}^*(\lambda)$ . Since the projection  $\pi : \mathfrak{g}^* \rightarrow \mathfrak{g}_0^*$  is invariant with respect to  $\text{Ad}_{G_0}^*$ , the map  $\pi$  project  $\Omega(\lambda)$  onto  $\omega$ .

It remains to show that

$$\Omega(\lambda) \cap \pi^{-1}(\lambda_0) = \{\lambda\} \quad (6)$$

for any  $\lambda \in \pi^{-1}(\lambda_0)$ . Suppose that  $\lambda' = \text{Ad}_g^* \lambda$  and  $\lambda, \lambda' \in \pi^{-1}(\lambda_0)$ . Since  $G = G_0 X$  and  $X \in G^\lambda$ , we verify that  $\lambda' = \text{Ad}_{g_0}^* \lambda$  for some  $g_0 \in G_0$ . As  $\lambda, \lambda' \in \pi^{-1}(\lambda_0)$ , the element  $g_0$  lies in stabilizer  $G_0^{\lambda_0}$ . Then  $g_0 = \exp(y_0)$  for some  $y_0 \in \mathfrak{g}_0^{\lambda_0}$ . We obtain

$$\lambda'(x) = \lambda(\text{Ad}_{\exp(-y_0)}^* x) = \lambda(x) - \lambda(\text{ad}_{y_0} x) + \sum_{k \geq 2} \frac{(-1)^k}{k!} \lambda_0(\text{ad}_{y_0}^k x). \quad (7)$$

Since  $x \in \mathfrak{g}^\lambda$ , we have  $\lambda(\text{ad}_{y_0} x) = 0$ . As  $y_0 \in \mathfrak{g}_0^{\lambda_0}$ , we have  $\lambda_0(\text{ad}_{y_0}^k x) = 0$  for any  $k \geq 2$ . Substituting into (7), we obtain  $\lambda'(x) = \lambda(x)$ . Using  $\lambda, \lambda' \in \pi^{-1}(\lambda_0)$ , we conclude  $\lambda = \lambda'$ .  $\square$

**Lemma 8.** For any subalgebra  $\mathfrak{h}$  of a nilpotent Lie algebra  $\mathfrak{g}$  there exists a chain of subalgebras  $\mathfrak{g} = \mathfrak{g}_0 \supset \mathfrak{g}_1 \supset \dots \supset \mathfrak{g}_k = \mathfrak{h}$  such that  $\mathfrak{g}_{i+1}$  is an ideal of codimension one in  $\mathfrak{g}_i$  for any  $1 \leq i \leq k-1$ .

**Proof.** We use the induction method for  $\dim \mathfrak{g}$ . For  $\dim \mathfrak{g} = 1$  the statement is obvious. Assume that the statement is true for  $\dim \mathfrak{g} = n-1$ ; let us prove it for  $n$ . The nilpotent Lie algebra  $\mathfrak{g}$  has a nonzero central element  $z$ . Consider projection  $\phi : \mathfrak{g} \rightarrow \bar{\mathfrak{g}} = \mathfrak{g}/Kz$ . The image  $\bar{\mathfrak{h}}$  is a subalgebra in  $\bar{\mathfrak{g}}$ . As  $\dim \bar{\mathfrak{g}} < n$ , according to induction assumption, there exists a chain of subalgebras  $\bar{\mathfrak{g}} = \bar{\mathfrak{g}}_0 \supset \bar{\mathfrak{g}}_1 \supset \dots \supset \bar{\mathfrak{g}}_k = \bar{\mathfrak{h}}$ , where  $\bar{\mathfrak{g}}_i$  is an ideal of codimension one in  $\bar{\mathfrak{g}}_{i+1}$  for any  $1 \leq i \leq k-1$ . Denote  $\mathfrak{g}_i = \phi^{-1}(\bar{\mathfrak{g}}_i)$ . If  $z \in \mathfrak{h}$ , then  $\mathfrak{g}_k = \mathfrak{h}$ ; this completes construction of the chain of subalgebras. If  $z \notin \mathfrak{h}$ , then  $\mathfrak{g}_k = \mathfrak{h} + Kz$ . It remains to put  $\mathfrak{g}_{k+1}$  equal to  $\mathfrak{h}$ .  $\square$

**Theorem 9.** Let  $G = \exp(\mathfrak{g})$  be a unipotent group over the finite field  $K$ ,  $\mathfrak{h}$  be a subalgebra of  $\mathfrak{g}$ ,  $H = \exp(\mathfrak{h})$ . Let  $\Omega$  (resp.  $\omega$ ) be a coadjoint orbit in  $\mathfrak{g}^*$  (resp.  $\mathfrak{h}^*$ ),  $T^\Omega$  and  $t^\omega$  be corresponding irreducible representations of  $G$  and  $H$ ,  $\pi$  be the natural projection  $\mathfrak{g}^*$  onto  $\mathfrak{h}^*$ . Denote  $m(\omega, \Omega) = \text{mult}(T^\Omega, \text{ind}(t^\omega, G)) = \text{mult}(t^\omega, \text{res}(T^\Omega, H))$ . Then

$$m(\omega, \Omega) = \frac{|\pi^{-1}(\omega) \cap \Omega|}{\sqrt{|\omega| \cdot |\Omega|}}.$$

**Proof.** Introduce notations

$$P = |\pi^{-1}(\omega) \cap \Omega|, \quad Q = \sqrt{|\omega| \cdot |\Omega|}, \quad M = \text{mult}(T^\Omega, \text{ind}(t^\omega, G)).$$

We shall prove that  $M = P/Q$  using the induction method with respect to  $\text{codim}(\mathfrak{h}, \mathfrak{g})$ . If  $\text{codim}(\mathfrak{h}, \mathfrak{g}) = 0$ , then  $\Omega = \omega$  and hence  $P = |\Omega|$ ,  $Q = |\Omega|$ ,  $M = 1$ ; this proves the equality  $M = P/Q$ .

Assume that the equality is proved for  $\text{codim}(\mathfrak{h}, \mathfrak{g}) < k$ ; let us prove for  $\text{codim}(\mathfrak{h}, \mathfrak{g}) = k$ . The lemma 8 implies that there exists a subalgebra  $\mathfrak{g}_1$  obeying the conditions  $\mathfrak{g} \supset \mathfrak{g}_1 \supset \mathfrak{h}$ ,  $\text{codim}(\mathfrak{h}, \mathfrak{g}_1) = 1$ . Choose  $\lambda_0 \in \omega$ . The natural projections  $\pi : \mathfrak{g}^* \rightarrow \mathfrak{h}^*$ ,  $\pi_1 : \mathfrak{g}^* \rightarrow \mathfrak{g}_1^*$ ,  $\pi_0 : \mathfrak{g}_1^* \rightarrow \mathfrak{h}^*$  satisfy  $\pi = \pi_0 \pi_1$ . For  $\mathfrak{g}_1^{\lambda_0} = \{x \in \mathfrak{g}_1 : \lambda_0[x, \mathfrak{h}] = 0\}$  only two cases are possible:  $\mathfrak{g}_1^{\lambda_0} \subset \mathfrak{h}$ , or  $\mathfrak{g}_1^{\lambda_0} \not\subset \mathfrak{h}$ . 1) Case  $\mathfrak{g}_1^{\lambda_0} \subset \mathfrak{h}$ . Following lemma 7,  $\pi_0^{-1}(\omega)$  belongs to the same coadjoint orbit  $\Omega_1 \subset \mathfrak{g}_1^*$  of the group  $G_1 = \exp(\mathfrak{g}_1)$ . Therefore,  $\dim \Omega_1 = \dim \omega + 2$  and

$$|\Omega_1| = q^2 |\omega|. \tag{8}$$

A polarization  $\mathfrak{p}_0$  for  $\lambda_0$  in  $\mathfrak{h}$  is also a polarization for any  $\lambda_1 \in \pi_0^{-1}$  in  $\mathfrak{g}_1$ . Really,  $\mathfrak{p}_0$  is an isotropic subspace in  $\mathfrak{g}_1$  and

$$\text{codim}(\mathfrak{p}_0, \mathfrak{g}_1) = \text{codim}(\mathfrak{p}_0, \mathfrak{h}) + 1 = \frac{1}{2}(\dim \omega + 2) = \frac{1}{2} \dim \Omega_1.$$

The induced representation  $\text{ind}(t^\omega, G_1)$  is irreducible and coincides with  $T^{\Omega_1}$ ,

$$\text{ind}(T^{\Omega_1}, G) = \text{ind}(t^\omega, G). \quad (9)$$

According to the induction assumption,

$$\text{mult}(T^\Omega, \text{ind}(T^{\Omega_1}, G)) = \frac{|\pi^{-1}(\Omega_1) \cap \Omega|}{\sqrt{|\Omega_1| \cdot |\Omega|}}. \quad (10)$$

Applying the formula (9), we obtain

$$M = \frac{|\pi^{-1}(\Omega_1) \cap \Omega|}{\sqrt{|\Omega_1| \cdot |\Omega|}}. \quad (11)$$

Using (8), we conclude

$$M = \frac{qP}{\sqrt{q^2|\omega| \cdot |\Omega|}} = \frac{P}{Q}. \quad (12)$$

2) Case  $\mathfrak{g}_1^{\lambda_0} \not\subset \mathfrak{h}$ . By the formula (6) we obtain

$$\pi^{-1}(\omega) = \bigcup_{\lambda_1 \in \pi_0^{-1}(\lambda_0)} \pi_1^{-1}(\Omega_1(\lambda_1)),$$

where  $\Omega_1(\lambda_1)$  is an orbit of  $\lambda_1 \in \mathfrak{g}^*$  with respect to  $\text{Ad}_{G_1}^*$ . Applying the induction assumption, we obtain

$$\begin{aligned} P = |\pi^{-1}(\omega) \cap \Omega| &= \sum_{\lambda_1 \in \pi_0^{-1}(\lambda_0)} |\pi^{-1}(\Omega_1(\lambda_1)) \cap \Omega| = \\ &= \sum_{\lambda_1 \in \pi_0^{-1}(\lambda_0)} \sqrt{|\Omega_1(\lambda_1)| \cdot |\Omega|} \text{mult}(T^\Omega, \text{ind}(T^{\Omega_1(\lambda_1)}, G)). \end{aligned}$$

Since  $|\Omega_1(\lambda_1)| = |\omega|$ , we have

$$P = Q \sum_{\lambda_1 \in \pi_0^{-1}(\lambda_0)} \text{mult}(T^\Omega, \text{ind}(T^{\Omega_1(\lambda_1)}, G)). \quad (13)$$

From the other hand, for any polarization  $\mathfrak{p}_0$  of  $\lambda_0$  the subalgebra

$$\mathfrak{p} = \mathfrak{g}_1^{\lambda_0} + \mathfrak{p}_0$$



is a polarization for any  $\lambda_1 \in \pi_0^{-1}(\lambda_0)$ . Representation  $\text{ind}(\xi_{\lambda_0}, P_0, P)$  is a direct sum of one dimensional representations  $\xi_{\lambda_1}$ , where  $\lambda_1 \in \pi_0^{-1}(\lambda_0)$ . Therefore,  $\text{ind}(t^\omega, G)$  is a direct sum of representations  $\text{ind}(T^{\Omega_1(\lambda_1)}, G)$  where  $\lambda_1 \in \pi_0^{-1}(\lambda_0)$ . We obtain

$$M = \text{mult}(T^\Omega, \text{ind}(t^\omega, G)) = \sum_{\lambda_1 \in \pi_0^{-1}(\lambda_0)} \text{mult}(T^\Omega, \text{ind}(T^{\Omega_1(\lambda_1)}, G)). \quad (14)$$

Substituting (14) in (13), we verify  $P = QM$ .  $\square$

**Corollary 10.** The irreducible representation  $T^\Omega$  occurs in decomposition of  $\text{ind}(t^\omega, G)$  if and only if the orbit  $\Omega$  has a nonempty intersection with  $\pi^{-1}(\omega)$ .

**Corollary 11.** The irreducible representation  $t^\omega$  occurs in decomposition of the restriction of representation  $T^\Omega$  on the subgroup  $H$  if and only if the orbit  $\omega$  lies in  $\pi(\Omega)$ .

**Corollary 12.** Let  $\Omega, \Omega_1, \Omega_2$  be coadjoint orbits in  $\mathfrak{g}^*$ . Denote by  $|M|$  the number of elements in the subset

$$M = \{(\lambda_1, \lambda_2) : \lambda_1 \in \Omega_1, \lambda_2 \in \Omega_2, \lambda_1 + \lambda_2 \in \Omega\}.$$

Then

$$\text{mult}(T^\Omega, T^{\Omega_1} \otimes T^{\Omega_2}) = \frac{|M|}{\sqrt{|\Omega| \cdot |\Omega_1| \cdot |\Omega_2|}}.$$

**Proof.** We apply theorem 9 to the group  $G \times G$ , its coadjoint orbit  $\Omega_1 \times \Omega_2$ , subgroup  $H = \{(g, g) : g \in G\}$  and its orbit  $\omega = \{(\lambda, \lambda) : \lambda \in \Omega\}$ .  $\square$

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